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The stability of inviscid flows over passive compliant walls

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The linear temporal stability of incompressible semi-bounded inviscid parallel flows over passive compliant walls is studied. It is shown that some of the well-known classical results for inviscid parallel flows with rigid boundaries can, in fact, be extended in modified form to passive compliant walls. These include a result of Rayleigh (1880) which shows that the real part of the phase velocity of a non-neutral disturbance must lie within the range of the velocity distribution; the semi-circle theorem of Howard (1961) and a result of Høiland (1953) which places a bound on the temporal amplification rates of unstable disturbances. The bounds on the phase velocity and the temporal amplification rates of unstable two-dimensional disturbances provide useful guides for numerical studies.

The results are valid for a large class of passive compliant walls. This generality is achieved through a variational-Lagrangian formulation of the essential dynamics of wall motion. A general treatment of the marginal stability of thin shear flows over general passive compliant walls is given. It represents a generalization of the analysis given by Benjamin (1963) for membrane and plate surfaces. Sufficient conditions for the stability of thin shear flows over passive compliant walls are deduced. The applications of the stability criteria to simple cases of compliant wall are described to illustrate the use and the effectiveness of these criteria.

1. Introduction

Historically, interest in the stability of flows over compliant walls can be said to have begun with the pioneering experimental work of Kramer (1960). In a series of experiments conducted in Long Beach Harbour, California, Kramer reported that by coating his streamlined models with a special compliant coating he was able to achieve a dramatic reduction in the total drag on the models. Reductions of up to 60 % over rigid models of equivalent displacement were obtained. Kramer attributed this reduction in the drag to a delaying in the transition of the boundary layer to a state of turbulence. Kramer's fundamental thesis was that the compliant response of his coating and the viscous dissipation within the coating had damped out the Tollmien–Schlichting instability waves which are known to be the precursor of boundary-layer transition over rigid surfaces. The quest for drag reduction has remained ever since a prime motivation for subsequent studies, experimental and theoretical, by other investigators. The reduction in self-noise attendant upon the suppression of flow turbulence is another practical incentive.

An important part of our understanding of the fundamental aspects of the stability of the coupled flow-wall system stems from the earlier theoretical studies, notably those of Benjamin (1960, 1963) and Landahl (1962) among others. The theoretical study of Benjamin (1960) revealed that the stability problem of the interaction between the flow and the wall is far more complex than Kramer had initially envisaged. Benjamin found that the Tollmien–Schlichting waves are stabilized by compliant wall response but destabilized by viscous dissipation within the wall. This latter discovery ran counter to Kramer's hypothesis. In addition to the Tollmien–Schlichting instability, Benjamin's study also uncovered the existence of other instabilities; those that are associated with the resonant modes of the compliant wall and a Kelvin–Helmholtz-type instability. Unlike the Tollmien–Schlichting instability, these latter instabilities can exist even when the flow is inviscid. And it is the instabilities that can arise from the interaction between an inviscid flow and a passive compliant wall that is of principal concern here. By the adjective passive is meant a wall which does not possess within itself any source of energy which may be a cause of instability. Inviscid flows over a compliant wall may also suffer from another instability which assumes the form of a slowly travelling instability wave. This is frequently termed the static divergence instability. Recently Carpenter & Garrad (1986) considered an example which indicates that the static divergence instability may well be an absolute instability, that is, a temporal instability having zero group velocity.

A useful energy-related classification of the instabilities that may arise from the interaction of the flow and the compliant wall was introduced by Benjamin (1963); drawing upon the work of Landahl (1962). In this scheme, the instabilities, termed Class A, Class B or Class C, are characterized by the sign of the energy needed to activate or excite the initial disturbances. In the case of Class A waves, they require the extraction of energy from the initially unperturbed system for activation and are consequently destabilized by dissipation. The Tollmien–Schlichting-type instability and the static-divergence instability fall within this class. Class B instability has the opposite characteristics. Instabilities associated with the resonant modes of the compliant walls are normally Class B and are stabilized by dissipation in the wall. The stability of these waves is determined by the net effect of irreversible processes which include dissipation and the transfer of energy by non-conservative hydrodynamic force generated by the critical-layer mechanism of Miles (1957). Class C instabilities, on the other hand, are those which involve mainly conservative transfer of energy from the flow to the wall so that the total energy of the system remains virtually unchanged – more correctly the activation energy because the total energy is always reduced by dissipation. The Kelvin–Helmholtz type of instability is a prime example of Class C instability.

Much of the early theoretical analyses were carried out on simple wall models such as those of membranes and plates. More recently a class of compliant walls modelled as plate supported on an elastic foundation and backed by a liquid substrate (inviscid and viscous) was studied by Carpenter and co-workers. The model is purported to represent the principal mechanical attributes of the Kramer walls and has been employed by them to study the possible stability performance of the original Kramer's walls; see Carpenter & Garrad (1982, 1985) for examples. The stability of inviscid parallel flows over an elastic or viscoelastic layer backed by a rigid base was also recently studied by Duncan, Waxman & Tulin (1985), Evrensel & Kalnin (1985) and Fraser & Carpenter (1985) among others. We will not go into a more complete discourse of the existing literature on inviscid flows over compliant walls. Instead due references to the works of other investigators will be made in the main text as and when they are relevant. The very recent article of Carpenter & Garrad (1986) and the further references contained therein are strongly recommended to those who wish to seek a more complete overview of the subject.

One aspect of the subject which appears to have been largely neglected in the literature is the examination of the status of the various classical theorems, which are an integral part of the linear stability theory of inviscid parallel flows with rigid boundaries, in relation to compliant walls. The studies known to the authors are those of Chin (1979) and Callan & Case (1981). Callan & Case concluded that most of the classical results cannot be carried over to compliant walls with the possible exceptions of the Rayleigh's inflexion-point theorem and Fjørtoft's theorem, both of which hold under very restrictive requirements on the boundary condition at the flow-wall interface. Their overall conclusions are not shared by the present authors and in this paper we establish that some of the important classical results can indeed be extended to passive compliant walls in modified form. Chin (1979) presented some analyses of the Rayleigh equation for the special case of flow over flexible membranes. His result will be subsequently discussed.

A crucial term which recurs frequently in this paper is

$$\hat{t}_3\,\hat{\eta}_3^*|_{x_3=0}, \tag{1.1}$$

where $\hat{t}_3$ and $\hat{\eta}_3$ are respectively the complex amplitudes of the vertical perturbation traction applied by the flow to the wall and the vertical displacement of the flow-wall interface (the meaning of the terms should become clearer in the main text). Superscript * denotes complex conjugation. This term is clearly related to the transfer of work at the flow-wall interface and is henceforth referred to herein as the work-transfer term. It embodies the essential ingredients of the interaction between the flow and the wall. In §2 we describe a general representation of the work-transfer term (1.1) in terms of certain characteristic quantities of the wall for the class of disturbances of the travelling-wave type that is admitted by the problem. A variational-Lagrangian approach to the derivation of the required representation which is applicable to a large class of passive compliant walls is outlined. To motivate the form of (1.1), we begin §2 with another approach applied to a single-layer viscoelastic wall. The approach, which amounts to the derivation of the energy-balance equation for the wall from its governing equations, can also be generalized to other types of compliant wall, but its formal generalization is less convenient.

Given the general representation for the work-transfer term at the flow-wall interface, §3 establishes three propositions which constitute the extensions of three classical results to passive compliant walls. The three classical results in question are, a result of Rayleigh which states that the real phase velocity c_r for an unstable disturbance must lie in the range of $U(x_3)$, the two-dimensional velocity profile; the semicircle theorem of Howard (1961) and a result of Høiland (1953) which places a bound on the temporal amplification rates of unstable disturbances. The validity of the extensions only requires that the relevant characteristic integrals of the wall (a kinetic-energy integral, a potential- or stored-energy integral and a dissipation integral) be positive and this is naturally fulfilled by most passive compliant walls subject to small initial equilibrium stresses. The bounds on the phase velocity provide useful guides for numerical studies by narrowing down the domain in the complex c-plane where unstable eigenvalues may exist. A simple application of the bounds on the real phase velocity c_r is considered in §4.

In §4, the theory developed in §§2 and 3 is applied to the consideration of the stability of thin shear flows over passive compliant walls in general. Conditions of marginal instability which represent a generalization of the analysis given by Benjamin (1963) for membrane and plate surfaces are briefly described. Sufficient conditions for marginal stability (in the context of linear theory) which assume the form of a free-wave criterion and a static criterion are also deduced based largely upon well-established variational principles. The generality and the usefulness of these criteria are illustrated for simple compliant wall models. The results obtained accord well with those of Carpenter and co-workers and others. Comparisons with recent experimental results on turbulent flows over compliant walls are made.

2. A general description of wall motion

Figure 1 shows an inviscid semi-bounded shear flow over a passive compliant wall which is composed of a single layer of viscoelastic material backed by a rigid base. The standard coordinate frame (x_1, x_2, x_3) employed here is a right-handed Cartesian frame which has the positive x_1-axis pointing in the streamwise direction from left to right. The unperturbed surface of the wall spans the (x_1, x_2)-plane and the positive x_3-axis is outwardly normal to the surface. The compliant wall is subject to a two-dimensional sinusoidal travelling wave perturbation having a x_1-wavelength of $\lambda(=2\pi/\alpha)$ where α is the x_1-wavenumber of the perturbation. The amplitude of the wave is assumed to be an infinitesimal fraction of the wavelength but has been exaggerated in figure 1 for expositional clarity.

The primary objective of this section is the derivation of a generalized representation of the work-transfer term (1.1) which is valid for a large class of passive compliant walls. To begin with, the computation of the work-transfer term is illustrated for the case of the single uniform layer of viscoelastic material shown in figure 1. The form assumed by the work-transfer term is then generalized to other compliant walls via a variational-Lagrangian formulation. For simplicity of notation we assumed that the quantities of the walls are defined in the same lengthscale and velocity scale as the external flow. The density of the fluid flow is taken to be the reference density for both.

2.1. *A single-layer viscoelastic wall*

The non-dimensional local equations of motion for the wall layer in the absence of body forces are given by

$$\rho \ddot{\eta}_i = \sigma_{ij,j} \quad (i = 1, 2, 3), \tag{2.1}$$

where ρ is the density of the wall relative to flow density, $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)^{\mathrm{T}}$ is the displacement vector of a Lagrangian or material coordinate frame and $[\sigma_{ij}]$ is the stress tensor. Each dot above $\boldsymbol{\eta}$ denotes the partial derivative $\partial/\partial t$ with respect to time. Here the summation convention for repeated subscripts or superscripts is assumed and $(\ )_{,j}$ denotes partial derivative of $(\ )$ with respect to x_j. The assumption of zero body forces holds throughout the paper. Equations (2.1) can be multiplied by $\dot{\eta}_i^*$ and combined to give

$$\rho \ddot{\eta}_i \dot{\eta}_i^* = (\dot{\eta}_i^* \sigma_{ij})_{,j} - \dot{e}_{ij}^* \sigma_{ij}, \tag{2.2}$$

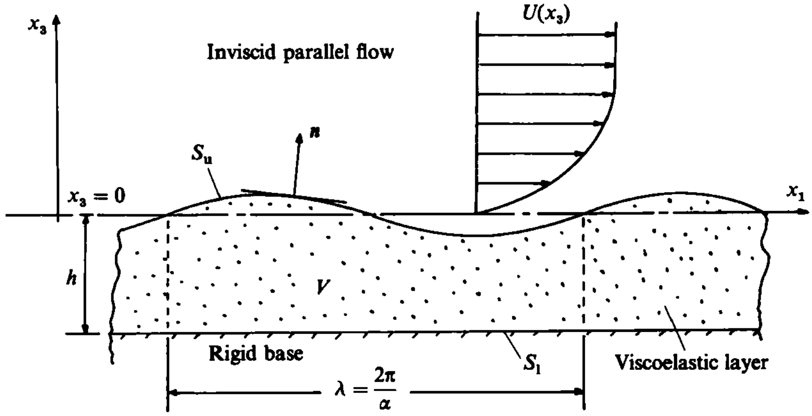

FIGURE 1. Inviscid parallel flow over a passive compliant wall.

where $[\epsilon_{ij}]$ is the linear strain tensor. The superscript * stands for complex conjugation. The integration of (2.2) over the volume V spanned by one wavelength λ of the disturbance in the x_1 direction and unit width in the x_2 direction and the application of the divergence theorem yield

$$\int_V \rho \ddot{\eta}_i \dot{\eta}_i^* \,\mathrm{d}V = \int_{S_u \cup S_l} \dot{\eta}_i^* \sigma_{ij} n_j \,\mathrm{d}S - \int_V \dot{\epsilon}_{ij}^* \sigma_{ij} \,\mathrm{d}V, \tag{2.3}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)^{\mathrm{T}}$ is the unit normal to the boundaries.

For 'almost reversible' deformation in a material with viscous damping, the constitutive relation between the stress tensor and the strain and strain-rate tensors has the general form (see §34 of Landau & Lifshitz 1970)

$$\sigma_{ij} = c_{ijkl}\,\epsilon_{kl} + d_{ijkl}\,\dot{\epsilon}_{kl}, \tag{2.4}$$

where $[c_{ijkl}]$ and $[d_{ijkl}]$ are fourth-order elastic modulus and viscosity tensors. Both $[c_{ijkl}]$ and $[d_{ijkl}]$ are real and have the usual symmetry properties. In general, they are a function of frequency. The substitution of (2.4) into (2.3) gives

$$\int_V \rho \ddot{\eta}_i \dot{\eta}_i^* \,\mathrm{d}V = \int_{S_u \cup S_l} \dot{\eta}_i^* \sigma_{ij} n_j \,\mathrm{d}S - \int_V c_{ijkl}\,\dot{\epsilon}_{ij}^* \epsilon_{kl} \,\mathrm{d}V - \int_V d_{ijkl}\,\dot{\epsilon}_{ij}^* \dot{\epsilon}_{kl} \,\mathrm{d}V. \tag{2.5}$$

The real part of (2.5) is the energy-balance equation associated with the propagating disturbance over one wavelength of the wall.

By the linearity of the governing equations for the disturbances in both the flow and the wall layer, and the geometry of the problem, the displacement vector field $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)^{\mathrm{T}}$ has a separable x_1 travelling wave solution of the form

$$\eta_j = q(t)\,\psi_j(x_1, x_2, x_3), \tag{2.6}$$

where
$$\psi_j = \chi(x_1)\,\hat{\eta}_j(x_3) \quad (j = 1, 2, 3),$$
$$q(t) = \mathrm{e}^{-\mathrm{i}\omega t} \quad (\omega = \alpha c), \quad \chi(x_1) = \mathrm{e}^{\mathrm{i}\alpha x_1},$$

and $\hat{\eta}_j(x_3)$ is the x_3 dependent complex amplitude of the displacement component η_j. In the case of temporal stability the x_1 wavenumber α is real while ω and c are, in general, complex. There is instability when ω_i is positive.

Using (2.6), the time-dependent strain tensor can be factored as

$$\epsilon_{kl} = q e_{kl}, \tag{2.7a}$$

where $e_{kl} = \frac{1}{2}(\psi_{k,l} + \psi_{l,k})$. The surface-traction vector $\boldsymbol{t} = (t_1, t_2, t_3)^{\mathrm{T}}$, given by

$$t_i = \sigma_{ij} n_j, \tag{2.7b}$$

has the separable form $\boldsymbol{t} = q\chi\hat{\boldsymbol{t}}$. Substituting (2.6) and (2.7) into (2.5) yields, after some algebraic manipulation,

$$\hat{t}_i \hat{\eta}_i^* |_{\mathrm{w}} + \hat{t}_i \hat{\eta}_i^* |_{x_3=-h} = -2\alpha^2 c^2 I + 2E - 2\mathrm{i}\alpha c D, \tag{2.8a}$$

where
$$I = \frac{1}{2\lambda} \int_V \rho \psi_j \psi_j^* \,\mathrm{d}V, \tag{2.8b}$$

$$E = \frac{1}{2\lambda} \int_V c_{ijkl}\, e_{ij}^* e_{kl} \,\mathrm{d}V, \tag{2.8c}$$

$$D = \frac{1}{2\lambda} \int_V d_{ijkl}\, e_{ij}^* e_{kl} \,\mathrm{d}V. \tag{2.8d}$$

The subscript w denotes evaluation at $x_3 = 0$.

In (2.8) the integral I is real and positive. The integrals E and D are real because of the symmetry properties of the tensors $[c_{ijkl}]$ and $[d_{ijkl}]$. D the dissipation integral will be positive because the integrand is a positive-definite quadratic form proportional to entropy production. The integral E, which is a measure of the average stored energy in the layer, will also be positive if the tensor $[c_{ijkl}]$ is defined with reference to the 'natural state' of zero stress. The terms on the left-hand side of (2.8a) have precisely the form of the work-transfer term (1.1), and we shall also refer to them as work-transfer terms.

The above derivation of (2.8) can be extended to cases consisting of any finite number of uniform viscoelastic layers. Provided continuity of displacements and stresses exists across the interfaces between adjacent layers, the boundary terms of adjacent layers at their common interfaces cancel. The domain of integration of the integrals I, E and D is then extended to cover all the relevant layers, and the boundary term $\hat{t}_i \hat{\eta}_i^* |_{x_3=-h}$ in (2.8) is replaced by evaluation at the lower surface of the last layer.

2.2. *A variational-Lagrangian formulation*

The derivation of (2.8) can be easily generalized to other types of compliant walls by adopting a variational-Lagrangian formulation for the dynamics of the wall. The usual starting point for such a formulation is the well-known Principle of Virtual Work of classical mechanics (see Washizu 1982). An important extension of this principle to cover irreversible thermodynamics was developed under the name of the Principle of Virtual Dissipation by M. A. Biot. This principle gives a more complete treatment of internal material dissipation. A comprehensive account of the principle and its applications to various continuous dynamical systems is given in the recent article of Biot (1984). Sections 12 (on viscoelastic solid) and 18 (on linear thermodynamics near equilibrium state) are of particular relevance to the generalization we seek. The finer points of the principle need not concern us here because of the assumed absence of any thermal effects and we shall only require the final result in the form of the Lagrangian equations.

Let $\eta = q\psi$ and its complex conjugate $\eta^* = q^*\psi^*$ be the solutions of the wall motion. Assuming that η and η^* are solution functions, q and q^*, both functions of time, can be treated as independent generalized variables. Calculus of variation method can then be applied via the mentioned principle to show that q and q^* satisfy the following Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{T}}{\partial \dot{q}}\right) - \frac{\partial \mathscr{T}}{\partial q} + \frac{\partial \mathscr{U}}{\partial q} + \frac{\partial \mathscr{D}}{\partial \dot{q}} = Q, \tag{2.9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{T}}{\partial \dot{q}^*}\right) - \frac{\partial \mathscr{T}}{\partial q^*} + \frac{\partial \mathscr{U}}{\partial q^*} + \frac{\partial \mathscr{D}}{\partial \dot{q}^*} = Q^*, \tag{2.9b}$$

where $\mathscr{T}$, $\mathscr{U}$ and $\mathscr{D}$ are respectively the average kinetic energy, the average potential or stored energy and the average dissipation integrals per unit length of the wall in the x_1-direction. For quasi-reversible processes about an equilibrium state, the dissipation integral $\mathscr{D}$ has the form

$$\mathscr{D} = \frac{1}{\lambda}\int_V D^V \,\mathrm{d}V, \tag{2.10a}$$

where D^V is a positive-definite quadratic form in the strain rates $(\dot{e}_{kl})_\mathrm{r}$ (or other appropriate deformation rates) such that

$$(\sigma^V_{ij})_\mathrm{r} = \frac{\partial D^V}{\partial (\dot{e}_{ij})_\mathrm{r}}, \tag{2.10b}$$

where $[\sigma^V_{ij}]$ is the viscous stress tensor associated with material viscous dissipation (see Biot 1984, §12). The generalized mechanical forces Q and Q^* on the boundaries are given by

$$Q\delta q + Q^*\delta q^* = \frac{1}{\lambda}\int_{S_\mathrm{u}\,\cup\, S_1} (t_j)_\mathrm{r}(\delta\eta_j)_\mathrm{r}\,\mathrm{d}S. \tag{2.11}$$

The subscript r denotes the real part.

2.3. *Applications of Lagrangian generalization*

We can now apply the Lagrangian formulation to derive (2.8) and we begin with the single-layer viscoelastic wall of figure 1. For this case, the kinetic energy integral $\mathscr{T}$ is given by

$$\begin{aligned}\mathscr{T} &= \frac{1}{2\lambda}\int_V \rho(\dot{\eta}_i)_\mathrm{r}(\dot{\eta}_i)_\mathrm{r}\,\mathrm{d}V \\ &= \frac{1}{4\lambda}\dot{q}\dot{q}^*\int_V \rho\psi_j\psi_j^*\,\mathrm{d}V = \tfrac{1}{2}\dot{q}\dot{q}^* I.\end{aligned} \tag{2.12}$$

The stored energy density is given by $\frac{1}{2}c_{ijkl}(\epsilon_{ij})_\mathrm{r}(\epsilon_{kl})_\mathrm{r}$ and the integral $\mathscr{U}$ is

$$\begin{aligned}\mathscr{U} &= \frac{1}{2\lambda}\int_V c_{ijkl}(\epsilon_{ij})_\mathrm{r}(\epsilon_{kl})_\mathrm{r}\,\mathrm{d}V \\ &= \frac{1}{4\lambda}qq^*\int_V c_{ijkl}\,e_{ij}\,e^*_{kl}\,\mathrm{d}V = \tfrac{1}{2}qq^* E.\end{aligned} \tag{2.13}$$

The local dissipation function D^V is obviously $\frac{1}{2}d_{ijkl}(\dot{e}_{ij})_r(\dot{e}_{kl})_r$ so that the integral $\mathscr{D}$ is

$$\mathscr{D} = \frac{1}{2\lambda}\int_V d_{ijkl}(\dot{e}_{ij})_r(\dot{e}_{kl})_r \,\mathrm{d}V$$
$$= \tfrac{1}{2}\dot{q}\dot{q}^*D. \tag{2.14}$$

and the generalized force Q^* is given by

$$Q^* = \tfrac{1}{4}q(\hat{t}_i\,\hat{\eta}_i^*|_{x_3=0}+\hat{t}_i\,\hat{\eta}_i^*|_{x_3=-h}), \tag{2.15}$$

where Q^* is the complex conjugate of Q. The substitution of (2.12)–(2.15) into (2.9b) yields (2.8) identically. The quadratic forms in terms q and q^* or $\dot{q}$ and $\dot{q}^*$ assumed by the integrals $\mathscr{T}$, $\mathscr{U}$ and $\mathscr{D}$ are of the forms given in (18.17)–(18.19) of Biot (1984). They are completely general and we can indeed use (2.12)–(2.14) as the basis to define the integrals I, E and D.

The analysis can also be extended to include incompressible inviscid or viscous fluid sublayers which have null basic velocity field (the topmost layer in contact with the flow being a solid layer). Let $\boldsymbol{u} = (u_1, u_2, u_3)^{\mathrm{T}} = \dot{q}(t)(\psi_1, \psi_2, \psi_3)^{\mathrm{T}}$ be the Eulerian perturbation velocity field of a viscous fluid layer of viscosity μ occupying a region V_1 extending over one wavelength λ in the x_1-direction. For an incompressible flow, the stored energy over one wavelength is zero in the absence of body forces. The integrals $\mathscr{T}$ and $\mathscr{D}$ are as given below

$$\mathscr{T} = \frac{1}{2\lambda}\int_{V_1}\rho(\boldsymbol{u})_r^2\,\mathrm{d}V, \quad \mathscr{D} = \frac{\mu}{\lambda}\int_{V_1}(\dot{e}_{ij})_r(\dot{e}_{ij})_r\,\mathrm{d}V, \tag{2.16a,b}$$

where $\dot{e}_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$. The corresponding quantities I and D are

$$I = \frac{2}{\dot{q}\dot{q}^*}\mathscr{T} = \frac{1}{2\lambda}\int_{V_1}\rho\psi_j\psi_j^*\,\mathrm{d}V, \quad D = \frac{2}{\dot{q}\dot{q}^*}\mathscr{D} = \frac{\mu}{\lambda}\int_{V_1}e_{ij}e_{ij}^*\,\mathrm{d}V, \tag{2.17a,b}$$

both of which are positive. $\mathscr{U}$ and hence also E are zero for an incompressible fluid.

We next go on to a frequently used idealized compliant-wall model of an elastic plate on a damped-elastic foundation as shown in figure 2, where m is the mass per unit area of plate surface, M is the mass moment of inertia about neutral axis, R is the flexural rigidity, k_{F} is the foundation elastic constant, $\sigma_{33} = k_{\mathrm{F}}\,\eta_3$ and k_μ is the foundation damping constant, $\sigma_{33}^V = k_\mu\,\dot{\eta}_3$. Equation (2.8) also holds for this compliant wall with I, E and D given by

$$I = \tfrac{1}{2}(m+M\alpha^2)\,|\hat{\eta}_3|^2, \tag{2.18a}$$

$$E = \tfrac{1}{2}(k_{\mathrm{F}}+R\alpha^4)\,|\hat{\eta}_3|^2, \tag{2.18b}$$

$$D = \tfrac{1}{2}k_\mu|\hat{\eta}_3|^2. \tag{2.18c}$$

Again we note that I, E and D are also positive real quantities.

In general, the work-transfer term at S_1 in (2.8) is zero for passive compliant walls that are terminated below by an interface on which either the displacements or the stresses or a combination of both are zero. Henceforth, we shall always use (2.8) with only the work-transfer term at $x_3 = 0$, the other term being assumed to be zero.

2.4. *Other remarks*

A Lagrangian representation of the wall dynamics had been used before by Benjamin (1963) as part of a more complete Lagrangian representation of the interactions of

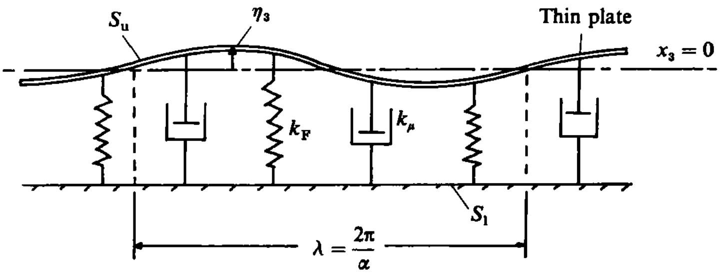

FIGURE 2. Thin plate on a damped elastic foundation.

a pair of coupled dynamical systems. It was within this larger Lagrangian framework that Benjamin (1963) developed his three-fold classification of the instabilities. The uses to which the Lagrangian representation for the wall dynamics are put in the present study are, however, entirely different from those of Benjamin (1963) who had as his primary aim the illustration of the very general nature of his classification. His terms μ, λ and κ correspond respectively to our I, E and D. He assumed his quantities μ, λ and κ to be constants in his analyses, although it is not clear if the assumption is strictly necessary. Our equivalents, the I, E and D are, of course, in general, functions of both c and α. For the more specific case of a membrane surface, Benjamin (1963) reverted back to non-Lagrangian treatment.

The general approach outlined in the two preceding sections shows that (2.8) applies to a wide category of passive compliant walls. These essentially consist of those passive walls which admit separable solutions of the travelling-wave form as given by (2.6). The quantities I and D are always positive and real. E will necessarily be positive when the elastic properties are assumed to be specified with reference to the natural state of zero stress. However, most systems in initial equilibrium are subject to some initial stresses. Initial stresses can be induced by body forces such as gravity for example, or boundary forces exerted by the basic flow. The presence of a significant initial stress state can theoretically make E negative because the elastic moduli tensor $[c_{ijkl}]$ need no longer be positive-definite. In the majority of wave-propagation problems, the initial stress state is customarily ignored or assumed to be inconsequential because of the great simplicity it affords to analyses. The omission is normally justifiable when the initial stress state is much less than the order of the material moduli. Cases with significant initial stress state are of very limited interest and relevance to the study because they do not, in general, admit separable solutions of the kind given by (2.6). Plane-wave solutions exist only in very special circumstances where the initial stress and deformation fields are homogeneous (see Eringen & Suhubi 1975, chapter 4). Hereinafter, E will be assumed to be greater than zero for non-trivial perturbations unless otherwise indicated.

3. The extension of the classical theorems

In the last section, the work-transfer term $\hat{t}_i \hat{\eta}_i^*$ was found to assume a common form, as given by (2.8), for a large class of passive compliant walls. In the present section, the work-transfer term at the flow-wall interface is related to the perturbation quantities of the flow domain. Three main propositions are proved.

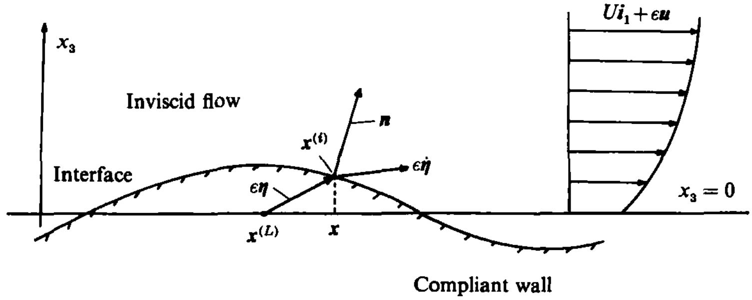

FIGURE 3. Flow–wall interface. $\boldsymbol{x} = (x_1, 0, 0)$; $\boldsymbol{x}^{(i)} = (x_1, 0, \epsilon\eta_3)$, coordinate of interface; $\boldsymbol{x}^{(L)} = (x_1^{(L)}, 0, 0)$, material coordinate of $\boldsymbol{x}^{(i)}$.

3.1. *The Rayleigh equation and the boundary conditions at the fluid–wall interface*

Let $U(x_3)$ be the basic velocity distribution of a two-dimensional incompressible inviscid parallel flow over a compliant wall as shown in figure 3. Let $\epsilon \boldsymbol{u} = (\epsilon u_1, 0, \epsilon u_3)^{\mathrm{T}}$ be a two-dimensional perturbation of the basic flow, where ϵ is a small real quantity.

If $\Phi = \mathrm{e}^{\mathrm{i}\alpha(x_1-ct)}\phi(x_3) = q(t)\ \chi(x_1)\,\phi(x_3)$ is the disturbance stream function such that

$$u_1 = \Phi_{,3}, \quad u_3 = -\Phi_{,1}, \tag{3.1}$$

then the complex amplitude $\phi(x_3)$ of the disturbance satisfies the well-known Rayleigh equation (see Drazin & Reid 1981)

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = 0, \tag{3.2}$$

where the superscript $'$ denotes the derivative with respect to x_3, $\mathrm{d}/\mathrm{d}x_3$. The Rayleigh equation (3.2) together with (i) $\phi \to 0$ as $x_3 \to \infty$, and (ii) the interface boundary conditions at the wall can be posed as an eigenvalue problem. In the case of temporal stability considered here, α is real and $c(= c_{\mathrm{r}} + \mathrm{i}c_{\mathrm{i}})$ is complex. The coupled system of flow and wall is deemed to be unstable when $\alpha c_{\mathrm{i}} > 0$ implying exponential growth in time. The system is said to be neutrally stable when $c_{\mathrm{i}} = 0$.

The interface boundary conditions are the continuity of normal velocity and stress at the interface. The continuity of normal velocity is given by (see figure 3)

$$\epsilon\dot{\boldsymbol{\eta}}\,|_{x^{(L)}}\cdot\boldsymbol{n} = (U\boldsymbol{i}_1+\epsilon\boldsymbol{u})\,|_{x^{(i)}}\cdot\boldsymbol{n}, \tag{3.3}$$

where $\boldsymbol{i}_1$ is the unit basis vector of the x_1 axis and $\boldsymbol{n}$ is the unit normal to the displaced interface at $\boldsymbol{x}^{(i)} = (x_1, 0, \epsilon\eta_3)$. The direction of the normal is given by the vector

$$\left(-\epsilon\frac{\partial\eta_3}{\partial x_1^{(L)}}, 0, 1+\epsilon\frac{\partial\eta_1}{\partial x_1^{(L)}}\right)^{\mathrm{T}},$$

where the superscript (L) is used to denote the material coordinate frame employed for the wall. When ϵ is very small, Taylor's expansion of (3.3) about $\boldsymbol{x}$ yields to order $O(\epsilon)$

$$\dot{\eta}_3+\frac{\partial\eta_3}{\partial x_1}U-u_3 = O(\epsilon), \tag{3.4}$$

at $\boldsymbol{x} = (x_1, 0, 0)$. The substitution of (2.6) and (3.1) gives

$$\hat{\eta}_3|_{\mathrm{w}} = \frac{-\phi_{\mathrm{w}}}{(U_{\mathrm{w}}-c)} \tag{3.5}$$

where we note that the subscript w denotes evaluation at $x_3 = 0$. By continuity, the perturbation stresses acting on the interface can be similarly linearized about $\boldsymbol{x}$ to give

$$\begin{aligned}\hat{\sigma}_{33}|_{\mathrm{w}} &= -\hat{p}_{\mathrm{w}} \\ &= (U_{\mathrm{w}}-c)\,\phi'_{\mathrm{w}} - U'_{\mathrm{w}}\,\phi_{\mathrm{w}},\end{aligned} \tag{3.6}$$

where $\hat{p}$ is the complex amplitude of the pressure perturbation p in the flow. There are no shear stress terms because of the inviscid nature of the flow. Equation (3.6) is obtained from the linearized x_1-momentum equation for the perturbation. Using (3.6) and (3.5), the work-transfer term

$$\begin{aligned}\hat{t}_j\,\hat{\eta}^*_j|_{\mathrm{w}} &= \hat{\sigma}_{33}\,n_3\,\hat{\eta}^*_3|_{\mathrm{w}} \\ &= -[(U_{\mathrm{w}}-c)\,\phi'_{\mathrm{w}} - U'_{\mathrm{w}}\,\phi_{\mathrm{w}}]\frac{\phi^*_{\mathrm{w}}}{(U_{\mathrm{w}}-c^*)},\end{aligned} \tag{3.7}$$

where it is noted that n_3 is $1+O(\epsilon)$.

Having established the relation between the work-transfer term and the flow perturbation quantities at the flow–wall interface at $x_3 = 0$, we can now proceed to extend the classical results.

3.2. *The propositions*

The proofs of the propositions follow that of the rigid-wall case. The important difference is that in the case of compliant walls, the boundary terms that result from the integration by parts need not be zero. The boundary term at $x_3 = \infty$ can normally be disregarded because admissible disturbances are assumed to vanish there. The presence of a non-zero boundary term at $x_3 = 0$ renders the classical results invalid for compliant walls. The crucial step in extending the results in a modified form to compliant walls is the identification of this boundary term with the work-transfer term at $x_3 = 0$.

PROPOSITION 1. *For an unstable wave, the real phase velocity c_r must satisfy*

$$U_{\mathrm{L}} < c_r < U_{\mathrm{U}}, \tag{3.8}$$

where $$U_{\mathrm{L}} = \min\,[U_{\min}, 0], \quad U_{\mathrm{U}} = \max\,[U_{\max}, 0],$$

and $$U_{\min} = \min_{x_3}\,[U(x_3)], \quad U_{\max} = \max_{x_3}\,[U(x_3)].$$

The bounds on c_r hold even when E is negative.

In proving the inequality in (3.8), we let

$$f = \frac{-\phi}{(U-c)}. \tag{3.9}$$

The Rayleigh equation (3.2) may be expressed in terms of f as

$$[(U-c)^2 f']' - \alpha^2 (U-c)^2 f = 0. \tag{3.10}$$

Multiply (3.10) by f^* and integrate over the flow domain from $x_3 = 0$ to $x_3 = \infty$. Integration by parts yields

$$\int_0^\infty (U-c)^2[|f'|^2 + \alpha^2 |f|^2]\,\mathrm{d}x_3 = -f^*_{\mathrm{w}}(U_{\mathrm{w}}-c)^2 f'_{\mathrm{w}}, \tag{3.11}$$

because by equation (3.9), $f(\infty) = \phi(\infty) = 0$. In the case of rigid boundary f_w is also zero and the right-hand side is zero. From (3.9) and (3.7), the work-transfer term at $x_3 = 0$ is

$$\hat{t}_j \hat{\eta}_j^* |_w = -f_w^*(U_w - c)^2 f_w'. \tag{3.12}$$

Using (3.12) and (2.8) in (3.11), we obtain

$$\int_0^\infty (U-c)^2 K \,\mathrm{d}x_3 = -\alpha^2 c^2 I + E - \mathrm{i}\alpha c D, \tag{3.13}$$

where $K(x_3) = \frac{1}{2}(|f'|^2 + \alpha^2 |f|^2)$ is positive-definite. The real and imaginary parts of (3.13) are respectively

$$\int_0^\infty [(U-c_r)^2 - c_i^2] K \,\mathrm{d}x_3 = -\alpha^2 (c_r^2 - c_i^2) I + E + \alpha c_i D, \tag{3.14}$$

and

$$2c_i \int_0^\infty (U - c_r) K \,\mathrm{d}x_3 = 2I\alpha^2 c_r c_i + \alpha c_r D. \tag{3.15}$$

Multiply (3.15) by $\frac{1}{2}\alpha$ to give

$$(\alpha c_i) \int_0^\infty (U - c_r) K \,\mathrm{d}x_3 = I\alpha^2 c_r (\alpha c_i) + \tfrac{1}{2}\alpha^2 c_r D. \tag{3.16}$$

The proof of the proposition can be obtained by an inspection of (3.16). We first consider the case for which $U_{\max}$ is greater than zero so that $U_U = U_{\max}$. Since I and D are real and positive, it follows immediately that for unstable waves, for which $\alpha c_i > 0$, c_r must not be greater than $U_{\max}$ to avoid contradiction in (3.16). If $U_{\max} < 0$, a contradiction of sign will again arise in (3.16) if $c_r \geqslant 0$. Thus for unstable waves, c_r must be less than $\max[U_{\max}, 0]$. An essentially symmetrical argument shows that c_r is bounded below by $\min[U_{\min}, 0]$. This proves the proposition. Since the term E, which is related to the stored energy, is not present in (3.16), the bounds on c_r also apply to cases with significant initial stresses, for which E may be negative.

In the case of a rigid wall, there are no right-hand-side terms in (3.16). When $\alpha c_i \neq 0$, c_r must necessarily lie in the range of $U(x_3)$; a result first noted by Rayleigh. The bounds in Proposition 1 differ from those of the rigid-wall case. For a compliant wall, there can exist unstable modes with $c_r < U_{\min}$ when $U_{\min}$ is greater than zero. An example that springs readily to mind is that of uniform flow over a compliant wall with $U_{\max} = U_{\min}$ greater than zero. The rigid-wall result of Rayleigh applies identically to the compliant-wall case only when $U_{\min}$ and $U_{\max}$ are of opposite signs or when at least one of either is zero. It is easy to deduce from the inequalities (3.8) that for unstable waves over passive compliant walls

$$|c_r| < \max[|U_{\min}|, |U_{\max}|]. \tag{3.17}$$

The next proposition is a modified extension of the semi-circle theorem of Howard (1961) to compliant walls.

Proposition 2. *For an unstable wave, the phase velocity c satisfies*

$$[c_r - \tfrac{1}{2}(U_U + U_L)]^2 + c_i^2 < [\tfrac{1}{2}(U_U - U_L)]^2, \tag{3.18}$$

where U_U and U_L are as defined in Proposition 1.

To prove the above proposition, we note that

$$\int_0^\infty (U - U_{\min})(U - U_{\max}) K \,\mathrm{d}x_3 \leqslant 0, \tag{3.19}$$

because K is positive-definite. Also from (3.14) and (3.15), we have

$$\int_0^\infty UK\,\mathrm{d}x_3 = \int_0^\infty c_\mathrm{r} K\,\mathrm{d}x_3 + \alpha^2 c_\mathrm{r} I + \frac{\alpha^2 c_\mathrm{r} D}{2\alpha c_\mathrm{i}}, \tag{3.20}$$

and

$$\int_0^\infty U^2 K\,\mathrm{d}x_3 = \int_0^\infty |c|^2 K\,\mathrm{d}x_3 + \alpha^2 |c|^2 I + E + \frac{\alpha^2 |c|^2 D}{\alpha c_\mathrm{i}}. \tag{3.21}$$

There are three cases to be considered, namely:

Case 1: $U_{\min} \leqslant 0$ and $U_{\max} \geqslant 0$,

Case 2: $U_{\min} > 0$ and $U_{\max} > 0$,

Case 3: $U_{\min} < 0$ and $U_{\max} < 0$,

A *reductio ad absurdum* type of argument is used to show that the inequality (3.18) holds in all the cases.

Case 1: $U_{\min} \leqslant 0$ and $U_{\max} \geqslant 0$.

For this case $U_\mathrm{L} = U_{\min}$ and $U_\mathrm{U} = U_{\max}$. Assuming (3.18) to be false, then its negation can be arranged in the form

$$|c|^2 - c_\mathrm{r}(U_{\min} + U_{\max}) + U_{\min} U_{\max} \geqslant 0. \tag{3.22}$$

The substitution of (3.20) and (3.21) into (3.19) yields

$$\int_0^\infty [|c|^2 - c_\mathrm{r}(U_{\min} + U_{\max}) + U_{\min} U_{\max}] K\,\mathrm{d}x_3 + I\alpha^2[|c|^2 - c_\mathrm{r}(U_{\min} + U_{\max})] + \frac{D\alpha^2}{\alpha c_\mathrm{i}}[|c|^2 - \tfrac{1}{2}c_\mathrm{r}(U_{\min} + U_{\max})] + E \leqslant 0. \tag{3.23}$$

The first two terms of (3.23) are positive by virtue of the assumption (3.22) and the fact that $U_{\min} U_{\max} \leqslant 0$. From the inequality (3.22), we have

$$|c|^2 - \tfrac{1}{2}c_\mathrm{r}(U_{\min} + U_{\max}) \geqslant \tfrac{1}{2}[U_{\min}(c_\mathrm{r} - U_{\max}) + U_{\max}(c_\mathrm{r} - U_{\min})], \tag{3.24}$$

the right-hand side of which is greater than zero by Proposition 1. The third term of (3.23) is hence positive. Since the fourth term E is also greater than zero, the sum of terms on the left-hand side of (3.23) is greater than zero leading to a contradiction with the right-hand side. The inequality (3.18) must therefore be true.

Case 2: $U_{\min} > 0$ and $U_{\max} > 0$.

For this case $U_\mathrm{L} = 0$ and $U_\mathrm{U} = U_{\max}$. The negation of the inequality (3.18) may be simplified to

$$|c|^2 - c_\mathrm{r} U_{\max} \geqslant 0. \tag{3.25}$$

Equation (3.23) can be recast as

$$\int_0^\infty [|c|^2 - c_\mathrm{r} U_{\max}] K\,\mathrm{d}x_3 + I\alpha^2[|c|^2 - c_\mathrm{r} U_{\max}] + \frac{D\alpha^2}{\alpha c_\mathrm{i}}[|c|^2 - \tfrac{1}{2}c_\mathrm{r} U_{\max}] + U_{\min}\left[\int_0^\infty (U_{\max} - c_\mathrm{r}) K\,\mathrm{d}x_3 - I\alpha^2 c_\mathrm{r} - \frac{D\alpha^2 c_\mathrm{r}}{2\alpha c_\mathrm{i}}\right] + E \leqslant 0. \tag{3.26}$$

The first two terms of (3.26) are positive by (3.25). Since c_r is positive by Proposition 1, the third term of (3.26) is also positive by (3.25). Since

$$\int_0^\infty (U_{\max}-U)\,K\,\mathrm{d}x_3 \geqslant 0,$$

the use of equation (3.20) shows the fourth term of (3.26) is also positive. The sum of the terms on the left-hand side of the inequality (3.26) is therefore greater than zero in contradiction with its right-hand side. The inequality (3.18) is thus true for this case.

Case 3: $U_{\min} < 0$ and $U_{\max} < 0$.

The argument for this case is closely similar to that for Case 2, but we shall illustrate it for completeness sake. In this case $U_L = U_{\min}$ and $U_U = 0$. The negation of the inequality (3.18) has the form

$$|c|^2 - c_r\,U_{\min} \geqslant 0. \tag{3.27}$$

Either of (3.23) and (3.26) can be rearranged to give

$$\int_0^\infty [|c|^2 - c_r\,U_{\min}]\,K\,\mathrm{d}x_3 + I\alpha^2[|c|^2 - c_r\,U_{\min}] + \frac{D\alpha^2}{\alpha c_i}[|c|^2 - \tfrac{1}{2}c_r\,U_{\min}]$$
$$+\,U_{\max}\left[\int_0^\infty (U_{\min}-c_r)\,K\,\mathrm{d}x_3 - I\alpha^2 c_r - \frac{D\alpha^2 c_r}{2\alpha c_i}\right] + E \leqslant 0. \tag{3.28}$$

The first two terms of (3.28) are positive by (3.27). The third term is positive by (3.27) and Proposition 1. Since

$$\int_0^\infty (U-U_{\min})\,K\,\mathrm{d}x_3 \geqslant 0,$$

the use of (3.20) shows that the fourth term is also positive. A contradiction hence arises in (3.28), and the inequality (3.18) must therefore be true. This completes the proof of Proposition 2.

As in the case of Proposition 1, the inequality (3.18) coincides with the original semicircle theorem for unstable waves when $U_{\min}$ and $U_{\max}$ are of opposite signs or when either of $U_{\min}$ or $U_{\max}$ is zero. In the other cases where $U_{\min}$ and $U_{\max}$ are of the same sign, the semicircle theorem of Howard is contained within the semicircle prescribed by (3.18); which has a radius of $\frac{1}{2}(U_U - U_L)$ and centred at $\frac{1}{2}(U_U + U_L)$ on the real axis of the complex c-plane.

We now prove the last proposition which gives a bound on the temporal amplification rate αc_i for basic velocity distributions which vanish at the wall.

Proposition 3. *For a basic velocity profile with $U_w = 0$, the temporal amplification rates αc_i of the unstable waves satisfy*

$$\alpha c_i < \tfrac{1}{2}\max_{x_3}[|U'(x_3)|]. \tag{3.29}$$

To prove this proposition, as in the rigid case (see Drazin & Howard 1966), we let

$$g = \phi/(U-c)^{\frac{1}{2}}, \tag{3.30}$$

where ϕ satisfies the Rayleigh equation (3.2). Substitution into (3.2) yields

$$[(U-c)\,g']' - g\left[\tfrac{1}{2}U'' + \alpha^2(U-c) + \frac{U'^2}{4(U-c)}\right] = 0. \tag{3.31}$$

Multiply (3.31) by g^* and integrate with respect to x_3 from 0 to ∞. Take the imaginary part and further multiply by c_i^2 to give

$$\int_0^\infty \left[\alpha^2 c_i^2 - \left(\frac{U' c_i}{2|U-c|}\right)^2\right] |g|^2 \,\mathrm{d}x_3 + c_i^2 \int_0^\infty |g'|^2 \,\mathrm{d}x_3 = -c_i \operatorname{Im}[c g'_w g^*_w]. \tag{3.32}$$

Here Im ( ) denotes the imaginary part of ( ). Since $c_i^2 \leqslant |U-c|^2$, (3.32) implies that

$$[\alpha^2 c_i^2 - \tfrac{1}{4}\max_{x_3}[|U'(x_3)|^2]] \int_0^\infty |g|^2 \,\mathrm{d}x_3 \leqslant -c_i \operatorname{Im}[c g'_w g^*_w]. \tag{3.33}$$

Using (3.30) and (3.7), the boundary term can be related to the work-transfer term,

$$\operatorname{Im}[c g'_w g^*_w] = -\frac{1}{|c|} \operatorname{Im}[c^* \hat{t}_j \hat{\eta}^*_j]_w. \tag{3.34}$$

The use of (2.8) and (3.34) in the inequality (3.33) finally results in

$$[\alpha^2 c_i^2 - \tfrac{1}{4}\max_{x_3}[|U'(x_3)|^2]] \int_0^\infty |g|^2 \,\mathrm{d}x_3 \leqslant -2\left[\alpha c_i |c| D + \alpha^2 c_i^2 |c| I + \frac{c_i^2}{|c|} E\right]. \tag{3.35}$$

The right-hand side of this inequality is less than zero for unstable waves, and therefore the inequality (3.29) holds.

In the rigid-wall case, the right-hand side of (3.35) is zero. The rigid-wall result was first established by Høiland (1953). Although the above propositions have been proved for a semi-bounded fluid domain, they are also, nevertheless, true when the flow is bounded above by a rigid boundary.

An alternative derivation of Propositions 1 and 2 employing the generalized Lagrangian-Mean formulation of Andrews & McIntyre (1978*a*, *b*) for the flow is outlined in Yeo (1986).

3.3. *On other classical results*

Having established the three propositions, we go on to consider the status of the other well-known classical results in relation to passive compliant walls. Of particular interest to us are those results that bear upon the roles of inflexion point in the basic flow profile. We shall confine ourselves to the simple case for which there is only one inflexion point.

The central role of inflexion point in the stability of parallel inviscid flows over rigid boundaries is epitomized by the famous inflexion-point theorem of Rayleigh (see Drazin & Reid 1981) which states that the existence of an inflexion point in the velocity profile is a necessary condition for flow instability. Equivalently, its absence is a sufficient condition for stability in the context of linear theory. When the boundary terms resulting from wall compliance are included, it is clear that this elegant result no longer holds. Flows deemed to be stable by Rayleigh's theorem are usually unstable when the boundary is sufficiently soft. From this generally, we see that wall compliance is destabilizing to inviscid flows. By the same token, Fjørtoft's (1950) stronger extension of Rayleigh's inflexion-point theorem is also invalid.

For monotonic velocity distributions, such as boundary-layer-type profiles over rigid wall, the critical point $x_3 = z_c$ (where $c = U(z_c)$) of a neutral mode can only occur at an inflexion point of the velocity distribution (see Lin 1955 or Drazin & Howard 1966). This is not the case in general for passive compliant walls. For a neutral mode, the Reynolds stress $\tau_R = -\overline{u_1 u_3}$ associated with the disturbance in the flow is

$$\tau_R = \tfrac{1}{2}\alpha \operatorname{Im}(\phi^* \phi'), \tag{3.36}$$

and τ_R is constant except at the critical point z_c where it suffers a discrete jump in its value of

$$\tfrac{1}{2}\alpha\pi\left(\frac{U_c''}{U_c'}\right)|\phi_c|^2, \tag{3.37}$$

from below to above the critical point (see Drazin & Reid 1981). The subscript c denotes evaluation at $x_3 = z_c$. Since τ_R must vanish at $x_3 = \infty$, for monotonic profiles the jump at z_c must be equal to the Reynolds stress developed at the mean position of the wall at $x_3 = 0$. From (3.36), (3.7) and (2.8),

$$\begin{aligned}\tau_R|_w &= -\tfrac{1}{2}\alpha\,\mathrm{Im}\,[\hat{t}_j\,\hat{\eta}_j^*]_w \\ &= \alpha^2 cD.\end{aligned} \tag{3.38}$$

Upon using (3.37) and (3.38), we have

$$2\alpha cD = -\pi\left(\frac{U_c''}{U_c'}\right)|\phi_c|^2, \tag{3.39}$$

which shows that for monotonic velocity distributions over passive compliant walls with non-zero dissipation ($D \neq 0$), the critical points of neutral modes cannot be points of inflexion. The exception being for the static case of $\alpha c = 0$. Tollmien's proof of the existence of unstable modes associated with inflexion points cannot, therefore, be extended to passive compliant walls with non-zero dissipation. This is not, however, to say that the existence of an inflexion point is not sufficient for instability. In fact, the evidence is that flows of monotonic velocity distribution with inflexion point(s) over dissipative compliant walls are always unstable. And it appears very likely that the existence of an inflexion point in flows with monotonic velocity profile is sufficient for instability.

The possible extension of classical results to general compliant walls was briefly investigated by Callan & Case (1981). They failed to obtain any really concrete results. On using the assumption that ϕ_w/ϕ_w' is real, they found that Rayleigh's inflexion theorem holds, and if further ϕ_w/ϕ_w' is negative as well, then Fjørtoft's theorem is also valid. These results, however, seem to be rather artificial because ϕ_w/ϕ_w' cannot realistically be assumed to be real for the two mentioned theorems which apply to non-neutral states. From (3.7) and (2.8) ϕ_w/ϕ_w' may only be real for neutral modes on non-dissipative walls ($D = 0$). For neutral modes, however, both Rayleigh's and Fjørtoft's theorems are not applicable. For the more specific compliant surface of a flexible membrane, Chin (1979) showed that the original semicircle theorem of Howard (1961) holds provided a certain inequality is satisfied. The inequality involves U_{max}, U_{min} and parameters of the surface. When U_{max} and U_{min} are of opposite signs, the said inequality is satisfied and his result is therefore consistent with our own modified semicircle theorem.

4. Applications

In this section, we apply the results of the preceding sections to derive general marginal stability criteria for thin shear flows over passive compliant walls with non-zero dissipation. The thin shear flow we have in mind is that which is a uniform flow with a very thin boundary layer over which the flow velocity $U(x_3)$ increases from zero at the undisturbed interface at $x_3 = 0$ and approaches the free-stream velocity U_∞ asymptotically. The boundary-layer thickness is assumed to be much less than the disturbance wavelength. The presence of the thin boundary layer enables non-conservative transfer of energy from the basic flow to take place when the disturbance wave is neutrally stable. Applications of the stability criteria to specific classes of compliant walls are also considered.

4.1. *The stability of thin shear flows*

From (3.7) and (2.8) we have

$$\hat{\sigma}_{33}|_w = \hat{\eta}_3|_w(E^{(n)} - \alpha^2 c^2 I^{(n)} - i\alpha c D^{(n)}), \tag{4.1a}$$

where

$$E^{(n)} = \frac{2E}{|\hat{\eta}_3|_w^2}, \quad I^{(n)} = \frac{2I}{|\hat{\eta}_3|_w^2}, \quad D^{(n)} = \frac{2D}{|\hat{\eta}_3|_w^2}. \tag{4.1b}$$

The superscript (n) indicates that the quantity denoted has been normalized by $\frac{1}{2}|\hat{\eta}_3|_w^2$. $E^{(n)}$, $I^{(n)}$ and $D^{(n)}$ are in essence $2E$, $2I$ and $2D$ respectively calculated over the time-independent displacement vector field $\boldsymbol{\psi}$ which has vertical displacement of unit amplitude at $x_3 = 0$. The stress amplitude $\hat{\sigma}_{33}|_w$ acting on the wall may be calculated from the integral of the x_3-perturbation momentum flow equation to give

$$\alpha^2 \int_0^\infty (U-c)\,\phi\,dx_3 = \hat{\eta}_3|_w(E^{(n)} - \alpha^2 c^2 I^{(n)} - i\alpha c D^{(n)}). \tag{4.2}$$

Equation (4.2) is the general stability equation for inviscid parallel flow over a passive compliant wall. To use (4.2), it is necessary to know how ϕ varies with α and c. For thin boundary-layer flow which has boundary-layer thickness much less than the wavelength of the disturbance, the small-α approximation of Drazin & Howard (1962) to ϕ may be employed,

$$\phi = A e^{-\alpha x_3} \sum_{j=0}^{\infty} \alpha^j H_j(x_3), \tag{4.3}$$

where A, a complex constant, and the H_j are determined by the boundary conditions on ϕ. It is not necessary to go into the details of how to find these quantities here. It suffices for our purpose merely to note that (4.3) is a uniformly valid small-α approximation of ϕ for a semi-bounded flow, and when the condition at infinity is applied, the α^1-approximation is given by

$$\phi(x_3) = A(U-c)\,e^{-\alpha x_3}\left\{1 + \alpha \int_{x_3}^{\infty}\left[\left(\frac{U_\infty - c}{U-c}\right)^2 - 1\right] dx_3\right\}. \tag{4.4}$$

This equation is identical to (7.3) of Benjamin (1959) which was fundamental to his task of calculating the boundary stresses induced by thin shearing flows over sinusoidal wavy surfaces. For the determination of neutral and near-neutral eigenstates of (4.1), we may thus adopt Benjamin's approximation to $\hat{\sigma}_{33}|_w$, which is given below,

$$\hat{\sigma}_{33}|_w \approx \hat{\eta}_3|_w\left[\alpha(U_\infty - c)^2 + i\alpha^2\pi(U_\infty - c)^4 \frac{U_c''}{(U_c')^3}\right] \tag{4.5}$$

for an inviscid flow; see also Benjamin (1963). Equation (4.5) is an approximation of $\hat{\sigma}_{33}|_w$ for a neutrally stable disturbance wave in a thin boundary-layer flow valid for small values of α. In this approximation $\hat{\sigma}_{33}|_w$ is immediately recognizable as the sum of the pressure perturbation in the uniform part of the flow and the out-of-phase contribution arising from the singularity at the critical point.

While being strictly an approximation for neutral modes, (4.5) may be employed for the study of incipient instability in the neighbourhoods of neutral states. The substitution of (4.5) into (4.1) yields a quadratic polynomial equation in c,

$$c^2(\alpha I^{(n)}+1)+c(\mathrm{i}\nu-2U_\infty)+(U_\infty^2-\alpha^{-1}E^{(n)})=0, \tag{4.6a}$$

where
$$\nu = D^{(n)}+\frac{\alpha\pi(U_\infty-c)^4U_c''}{c(U_c')^3}. \tag{4.6b}$$

Equation (4.6) governs the marginal stability of inviscid thin shear flow over a passive compliant wall. When $c(=c_r+\mathrm{i}c_i)$ is complex with a very small imaginary part, the terms U_c' and U_c'' in (4.6*b*) may be evaluated at the critical height z_c given by $U(z_c)=c_r$. Since $E^{(n)}$, $I^{(n)}$ and $D^{(n)}$ are all real, it is immediately obvious from (4.6) that ν must necessarily be zero for a neutral wave with the possible exception for the case when $c=0$. For very thin boundary layers, ϕ_c can be approximated by (7.25) of Benjamin (1959)

$$\phi_c = A\alpha(U_\infty-c)^2\,\mathrm{e}^{-\alpha z_c}/U_c', \tag{4.7}$$

with $A\approx\hat{\eta}|_w$ and $\mathrm{e}^{-\alpha z_c}\approx 1$. The substitution of (4.7) into (4.6*b*) shows that the vanishing of ν for neutral eigenstates is equivalent to the constancy of the Reynolds stress τ_R between the wall and the critical height, which earlier led to (3.39). Equations (3.39) and (4.6*b*) with $\nu=0$ are merely statements of energy conservation showing the balance between the non-conservative transfer of energy from the basic flow to the perturbation and its dissipation by the wall; $c\tau_R|_w$ being the average rate of work done on unit area of the wall by the flow. Because of the nature of the approximation for $\hat{\sigma}_{33}|_w$, (4.6) is applicable in the vicinity of neutral states and is consequently useful only for the determination of marginal stability conditions.

We now consider the stability implications of the marginal stability equation (4.6), which is valid for a large class of passive compliant walls. For a given wavenumber α, if c is an eigenvalue of the stability equation (4.6), then c must be at least one of the following two roots c_1 and c_2 of (4.6),

$$c_1 = \frac{(2U_\infty-A)-\mathrm{i}(\nu-B)}{2(\alpha I^{(n)}+1)}, \tag{4.8a}$$

$$c_2 = \frac{(2U_\infty+A)-\mathrm{i}(\nu+B)}{2(\alpha I^{(n)}+1)}, \tag{4.8b}$$

where
$$A=\tfrac{1}{2}(r+a)^{\frac{1}{2}},\quad B=\operatorname{sgn}(\nu)\,\tfrac{1}{2}(r-a)^{\frac{1}{2}}$$
$$(a-\mathrm{i}b)^{\frac{1}{2}}=A-\mathrm{i}B,\quad r^2=a^2+b^2,$$
$$a=4\alpha^{-1}E^{(n)}(\alpha I^{(n)}+1)-\nu^2-4\alpha U_\infty^2 I^{(n)},\quad b=4\nu U_\infty.$$

It is clear from (4.8) that $(c_1)_r\leqslant(c_2)_r$, $(c_1)_r<U_\infty$ and $(c_2)_r>0$. The wall quantities $E^{(n)}$, $I^{(n)}$ and $D^{(n)}$ are evaluated at the phase velocity c.

4.2. *General stability and instability criteria for passive compliant walls*

The conditions of stability or instability are usually obtained by analysing the imaginary parts of the roots of c. In the present case, we find that Proposition 1 of the last section may also be used to give sufficient conditions for stability.

By Proposition 1, a disturbance wave with $c_r<0$ or $c_r>U_\infty$ cannot be unstable. In particular, since we are concerned here only with marginal or threshold instability, there is no neutral state with $c_r<0$ or $c_r>U_\infty$ which is the limit-point (usual mathematical meaning) of unstable eigenstates; for otherwise it would imply the existence of unstable states with $c_r < 0$ or $c_r > U_\infty$. Neutral states with $c_r < 0$ or $c_r > U_\infty$, if they do exist are necessarily isolated from the unstable states and are consequently not marginal states as such. Thus to determine a condition or criterion for marginal stability, we merely need to find the condition under which the phase velocity c of neutral states cannot occur in the closed interval $[0, U_\infty]$. To ensure that the c do not lie in the interval for marginally stable states, we require that

$$c_1 < 0, \quad c_2 > U_\infty; \tag{4.9a, b}$$

noting that $c_1 \leqslant c_2$. At a given α, these inequalities are equivalent respectively to the following criteria for marginal stability,

$$U_\infty^2 < \frac{E^{(n)}(\alpha, c)}{\alpha}, \quad U_\infty^2 < \frac{E^{(n)}(\alpha, c)}{\alpha^2 I^{(n)}(\alpha, c)}. \tag{4.10a, b}$$

For closer comparison with the results of Benjamin (1963), we investigate the stability in the more conventional way by examining the imaginary parts of the roots of c given in (4.8). For the root c_1, there will be instability for $\nu > 0$ when $U_\infty^2 > E^{(n)}/\alpha$, which is Class A in Benjamin's (1963) classification. And it is not possible to have $\nu < 0$ because then $U_\infty^2 < E^{(n)}/\alpha$, and c_1 must be less than zero. If $c_1 < 0$, there is no critical point and ν cannot therefore be less than zero. For the root c_2, there will be instability when $\nu < 0$; termed Class B by Benjamin. However, ν can only be less than zero when $c_2 < U_\infty$ and for this we need $U_\infty^2 > E^{(n)}/(\alpha^2 I^{(n)})$. Obviously Class B instability can usually be suppressed by making $D^{(n)}$ sufficiently large so that ν becomes positive. This can, however, have the undesirable effect of destabilizing the Class A modes for which $\nu > 0$ is required for instability. When a in (4.8) is negative, $(c_1)_r = (c_2)_r$. There is then strong instability (termed Class C in Benjamin's classification) which sets in quite independently of the value of ν when $|\nu|$ is small, as it must at incipient condition. The condition for Class C instability is $a < 0$ which implies that $U_\infty^2 > E^{(n)}/\alpha + E^{(n)}/(\alpha^2 I^{(n)})$. This is normally not a crucial condition for incipient instability as the comparison with the onset conditions for Class A and Class B instabilities will show. It is the only class of instability that occurs in uniform flows over non-dissipative compliant walls. In summary, the conditions of instability corresponding to those discussed by Benjamin (1963) are, for

$$\text{Class A:} \quad U_\infty^2 > \frac{E^{(n)}}{\alpha}, \quad \nu > 0, \tag{4.11a}$$

$$\text{Class B:} \quad U_\infty^2 > \frac{E^{(n)}}{\alpha^2 I^{(n)}}, \quad \nu > 0, \tag{4.11b}$$

$$\text{Class C:} \quad U_\infty^2 > \frac{E^{(n)}}{\alpha} + \frac{E^{(n)}}{\alpha^2 I^{(n)}}, \tag{4.11c}$$

where ν is assumed to be very small in all cases. The quantity ν is directly related to Benjamin's net rate of irreversible energy conversion $\mathrm{d}E/\mathrm{d}t$ by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\tfrac{1}{2}\alpha^2 c^2 |\hat{\eta}_3|^2 \nu. \tag{4.12}$$

Equations (4.11) give the conditions for instability for thin-boundary-layer flows over passive compliant walls. The above treatment generalizes that of Benjamin (1963) for tensioned membranes and shows more clearly the roles played by the characteristic quantities $E^{(n)}$, $I^{(n)}$ and $D^{(n)}$ of the wall in determining the onset of instability. For any specific wall, the actual onset of instability can only be worked out with a knowledge of the basic velocity distribution as well as the properties of the wall.

For idealized passive compliant walls such as those of tensioned membranes and plates on damped elastic foundations, $E^{(n)}$, $I^{(n)}$ and $D^{(n)}$ have no dependence on the phase velocity c. For these walls, the marginal stability equation (4.6) yields only two eigenvalues for each value of wavenumber α. More generally, $E^{(n)}$, $I^{(n)}$ and $D^{(n)}$ have c-dependence which cannot be ignored. Such dependence on c usually results in an increase in the number of c-eigenvalues of (4.6) and this makes the study of stability more difficult. Also, any sufficient condition for stability must be able to take account of the multiplicities of eigenvalues. By the inequalities in (4.10), which are consequences of Proposition 1, there is no incipient instability when

$$U_\infty^2 < \min_{c_1, c_2}\left\{\frac{E^{(n)}(\alpha, c_1)}{\alpha}, \frac{E^{(n)}(\alpha, c_2)}{\alpha^2 I^{(n)}(\alpha, c_2)}\right\}, \tag{4.13}$$

where c_1 and c_2 vary over the eigenvalues of (4.6). However, (4.13) is quite useless as a stability criterion because the spectra of the c-eigenvalues are not normally known. If they were, there would be no problem to solve.

A simple way to determine a stability bound on U_∞ is to consider instead the minimization of the functionals $E^{(n)}/\alpha$ and $E^{(n)}/(\alpha^2 I^{(n)})$ with respect to the relevant classes of admissible displacement fields. Hence, for any given α, we may rewrite the bounds on U_∞ as

$$U_\infty^2 < \min_{\psi_1, \psi_2}\left\{\frac{E^{(n)}(\alpha, \psi_1)}{\alpha}, \frac{E^{(n)}(\alpha, \psi_2)}{\alpha^2 I^{(n)}(\alpha, \psi_2)}\right\}, \tag{4.14}$$

where ψ_1 and ψ_2 belong to the appropriate classes of admissible displacement fields. The inequality in (4.14) will normally represent a more stringent bound on U_∞ than (4.13) because the classes of admissible fields considered are, in general, larger than the class of eigenfunctions of the stability problem. In this form, it turns out that the minimum values of the functionals may be determined from well-established variational principles of solid mechanics as we show below.

To deduce the bound given by $\min_{\psi_2}\{E^{(n)}/(\alpha^2 I^{(n)})\}$ we note that

$$\frac{E^{(n)}(\alpha, \psi_2)}{I^{(n)}(\alpha, \psi_2)} = \alpha^2 c^2 = \omega^2, \tag{4.15a}$$

is the free-surface-wave eigenvalue problem for a non-dissipative compliant wall. The admissible displacement fields in this case are required to satisfy the necessary continuity and differentiability properties and prescribed geometric boundary conditions. By the Principle of Stationary Potential Energy (see for instance Washizu 1982), ω^2 is an eigenvalue of (4.15a) if and only if ω^2 is a stationary value of the quotient

$$\frac{E^{(n)}(\alpha, \psi_2)}{I^{(n)}(\alpha, \psi_2)}. \tag{4.15b}$$

The minimum value of (4.15a) is hence an eigenvalue of the free-wave eigenvalue problem. For a given wavenumber α, the minimum value of $E^{(n)}/I^{(n)}$ is the square of the lowest fundamental frequency which occurs with the wavelength $\lambda(=2\pi/\alpha)$ and we denote this lowest frequency by ω_α. The minimum value of $E^{(n)}/(\alpha^2 I^{(n)})$ is c_α^2 where $\omega_\alpha = \alpha c_\alpha$. The quantity c_α is merely the lowest free-surface wave speed for x_1-travelling waves with wavenumber α for the wall with an artificially enforced absence of dissipation. It is important to note that the dissipative quality of the wall plays no role in the above minimization of $E^{(n)}/(\alpha^2 I^{(n)})$.

We next consider the minimization of $E^{(n)}/\alpha$ and we begin by noting that $\frac{1}{4}\lambda E^{(n)}$ is the potential or stored energy in one wavelength of the wall corresponding to the admissible time-independent displacement field $\boldsymbol{\eta} = \boldsymbol{\psi}(\alpha, c)$. An admissible solution here is one which has the necessary continuity and differentiability properties and which satisfies the boundary conditions of the wall. By the Principle of Minimum Potential Energy (see Washizu 1982, §2.1 or Fung 1965), the value of $\lambda E^{(n)}$ is minimum for the unique admissible displacement field which corresponds to the solution of the problem of static equilibrium for the wall. The static problem in question may be taken to be the one which has a specification of unit-amplitude sinusoidal vertical displacement (corresponding to the imposed normalization $|\hat{\eta}_3| = 1$) and zero shear stress at $x_3 = 0$. In this case $E^{(n)} = 2E$. At the bottom of the compliant wall, we may have, depending on the case examined, either a specification of zero displacements or zero stresses or a combination of both. The unique solution of the static equilibrium problem is none other than the solution of the dynamical eigenvalue equation (4.1) in the limit as $c \rightarrow 0$. Since $I^{(n)}$ and $D^{(n)}$ must tend to zero as $c \rightarrow 0$, we are left with

$$\lim_{c\to 0}\left(\frac{\hat{\sigma}_{33}}{\hat{\eta}_3}\right)_{\mathrm{w}} = E^{(n)}, \tag{4.16}$$

and we denote the minimum value $\min_{\boldsymbol{\psi}_1}\{E^{(n)}/\alpha\}$ by $E_{\alpha}^{(n)}/\alpha$.

For a given α, a sufficient condition for the marginal stability of thin shear flows over passive compliant walls, in the context of linear stability theory, is hence given by

$$U_{\infty}^2 < \min\Big\{ \underset{\substack{\text{static}\\ \text{criterion}}}{E_{\alpha}^{(n)}/\alpha}, \quad \underset{\substack{\text{free-wave}\\ \text{criterion}}}{c_{\alpha}^2} \Big\}. \tag{4.17}$$

Equivalently, the negation of the inequality in (4.17) is a necessary condition for marginal instability. In a strict sense, the criterion in (4.17) is valid only for small values of the wavenumber α because the assumption of small α is implicit in the approximation to $\hat{\sigma}_{33}|_{\mathrm{w}}$ given by (4.5). The above derivation of the bounds on U_{∞} also sheds some light on the physical nature and the origin of the bounds. Comparison of the inequality (4.13) or (4.17) with the generalized criteria for instability in (4.11) indicates that the bound on U_{∞}^2 by c_{α}^2 corresponds precisely to the suppression of Class B instability. Similarly the bound on U_{∞}^2 by $E_{\alpha}^{(n)}/\alpha$ provides the necessary safeguard against Class A instability. In fact the sufficient condition for stability (4.13) could just as well have been deduced from the general criteria for instability by applying the necessary negations and reversing the implications. The use of Proposition 1 to derive the sufficient condition, however, does seem to be more straightforward and easier. The bound $U_{\infty}^2 < c_{\alpha}^2$ implies that there can be no Class B instability at wavenumber α when the lowest free-surface-wave speed of the wall (discounting dissipation) at the same wavenumber is greater than the free-stream velocity. From (4.16), it is obvious that the bound by $E_{\alpha}^{(n)}/\alpha = \lambda E_{\alpha}^{(n)}/(2\pi)$ is merely an assertion that there is no Class A instability when the wall is sufficiently stiff statically. From (4.16), $E_{\alpha}^{(n)}/\alpha$ is a measure of the vertical static stiffness of the wall to sinusoidal vertical stress distribution of wavelength λ. We shall refer to this static bound as the static criterion for stability and the other criterion as the free-wave criterion.

4.3. *Elastic plates on damped elastic foundations*

To deduce sufficient condition for the stability of thin boundary-layer flows over plates on damped elastic foundation we need to find the quantities $E^{(n)}$ and $I^{(n)}$. These can be obtained from (2.18) after due normalization. Using (2.18) and (4.1),

$$\frac{E^{(n)}_{\alpha}}{\alpha} = \frac{R\alpha^4 + k_F}{\alpha}, \quad c^2_{\alpha} = \frac{R\alpha^4 + k_F}{m\alpha^2}, \tag{4.18}$$

where we have neglected the mass moment of inertia term M for reason of algebraic simplicity. At wavenumber α, the sufficient condition for stability is hence given by

$$U^2_{\infty} < \min\left\{\frac{R\alpha^4 + k_F}{\alpha}, \frac{R\alpha^4 + k_F}{m\alpha^2}\right\}. \tag{4.19}$$

To extend this criterion over the widest possible range of α, bearing in mind the small α assumption, we can determine the minimum values of the terms in (4.18). $E^{(n)}_{\alpha}/\alpha$ has a minimum value of

$$4\left(\frac{k^3_F R}{27}\right)^{\frac{1}{4}} \quad \text{at } \alpha = \left(\frac{k_F}{3R}\right)^{\frac{1}{4}},$$

and c^2_{α} has a minimum value of

$$\frac{2}{m}(k_F R)^{\frac{1}{2}} \quad \text{at } \alpha = \left(\frac{k_F}{R}\right)^{\frac{1}{4}}.$$

The sufficient condition for stability for all α is thus that

$$U^2_{\infty} < \min\left\{4\left(\frac{k^3_F R}{27}\right)^{\frac{1}{4}}, \frac{2}{m}(k_F R)^{\frac{1}{2}}\right\}. \tag{4.20}$$

The sufficient condition $U^2_{\infty} < (2/m)(k_F R)^{\frac{1}{2}}$, which is the safeguard against Class B instability, is identical with that derived by Carpenter & Garrad (1986) for their travelling-wave-flutter (TWF) instability of thin boundary-layer flows over elastic plates on non-dissipative foundation (see their equation (3.7) for $T = 0$). In our case, it is not necessary to assume zero foundation damping. Indeed if there were to be no damping at all, the jump in Reynolds stress at the critical height z_c for the marginal state must be zero according to (3.39). The requirement of zero jump drastically restricts the possible values of c that a marginal state can have. This explains why in Carpenter & Garrad's case, the stability boundary is marked by $c = U_{\infty}$; that is in the free stream where $U'' = 0$.

The other part of the stability criterion $U^2_{\infty} < 4(k^3_F R/27)^{\frac{1}{4}}$ is also the same as that obtained by Carpenter & Garrad (see their equation (2.13)) for static divergence instability of uniform flow over the same wall in the presence of wall dissipation.

The above example clearly demonstrated the effectiveness of the very general stability criterion (4.17) derived in the last section.

4.4. *Single-layer viscoelastic isotropic walls*

As the next example, we determine the sufficient condition for the stability of an isotropic viscoelastic layer of thickness h perfectly bonded onto a rigid base. For simplicity, the material is assumed to be incompressible and to have a Voigt-deviatoric response to shear (see Bland 1960). The complex shear modulus is $G_s = G - \mathrm{i}\omega d$, where $G = \rho C^2_t$ is the elastic/storage modulus and ρ is the density of the material relative to the flow density. The quantity d is the material damping coefficient, and as has been noted before, d has no direct role in determining the bounds on U_∞.

By its very definition, c_α is determined by the elastodynamic characteristics of the wall. When the free-wave dispersion relation is known, c_α is the smallest c-eigenvalue of the relation at the wavenumber α. The free-wave dispersion relation for the wall can be found by solving the dynamic-wave-propagation problem for the wall, having set the material damping to zero, subject to zero surface traction along $x_3 = 0$. When $d \neq 0$, the free modes are all damped. The free-wave dispersion relation of the form given in (4.15a) is generally only useful for numerical computation purposes when there is some pre-knowledge of the mode shapes. For the single-layer wall in question, the dispersion relation required for the determination of c_α is given by

$$\tilde{\alpha}^2(4\tilde{\alpha}^2 b_T^2 + H^2)\tanh\tilde{\alpha}\tanh b_T - \tilde{\alpha} b_T(4\tilde{\alpha}^4 + H^2) + \frac{4\tilde{\alpha}^3 b_T H}{\cosh\tilde{\alpha}\cosh b_T} = 0, \tag{4.21}$$

where $$\tilde{\alpha} = \alpha h, \quad H = (\tilde{\alpha}^2 + b_T^2), \quad b_T^2 = \tilde{\alpha}^2\left(1 - \frac{c^2}{C_t^2}\right).$$

In (4.21), it can be seen that α and C_t only appear respectively in the combinations αh and c/C_t. It is useful to note that, except at rather small $\tilde{\alpha}$, the smallest c-eigenvalue at a given $\tilde{\alpha}$ is given by that branch of the eigenvalue which tends towards the Rayleigh free-surface wave speed c_{R} (of the elastic half-space problem) as $\alpha \rightarrow \infty$. On this Rayleigh branch, c is a monotonically decreasing function of $\tilde{\alpha}$. At small $\tilde{\alpha}$, the c-eigenvalues of all the branches are rather large. Over the complete range of $\tilde{\alpha}$, the Rayleigh wave speed c_{R} is necessarily the lowest free-wave speed that the walls support; see Gad-el-Hak, Blackwelder & Riley (1984) for example. Hence, c_α is bounded below by c_{R}, which in the case of an incompressible wall is given approximately by $0.9553C_t$. Thus, according to the free-wave criterion $U_\infty^2 < c_\alpha^2$, there is no Class B instability at any α when $U_\infty^2 < (c_{\mathrm{R}})^2 = 0.9126G/\rho$.

The static criterion for stability $U_\infty^2 < E_\alpha^{(\mathrm{n})}/\alpha$ is considered next. From earlier discussions, it is clear that $E^{(\mathrm{n})}$ can be determined in two ways. It can be found by solving the static equilibrium problem mentioned in §4.2. This can be accomplished quite routinely using the method of Love's strain functions (see Fung 1965). Alternatively, as suggested by equation (4.16), we can calculate the ratio $[\hat{\sigma}_{33}/\hat{\eta}_3]_{\mathrm{w}}$ in the limit as $c \rightarrow 0$ for the dynamic-wave-propagation problem. The repeated use of De l' Hopital rule is essential to overcome the singularities which result from the occurrences of coincident exponential solutions as $c \rightarrow 0$. Solution by either approaches is fairly straightforward but algebraically laborious. By either means we have

$$\frac{E_\alpha^{(\mathrm{n})}}{\alpha} = \frac{2G(\tilde{\alpha}^2 + \cosh^2(\tilde{\alpha}))}{\sinh\tilde{\alpha}\cosh\tilde{\alpha} - \tilde{\alpha}}, \tag{4.22}$$

which shows $E_\alpha^{(\mathrm{n})}/\alpha$ as a function of $\tilde{\alpha}$, and it has a minimum value of $2G$. Thus, regardless of the value of α, there is stability when

$$U_\infty^2 < \min\left\{\underset{\substack{\text{static}\\ \text{criterion}}}{2G}, \quad \underset{\substack{\text{free-wave}\\ \text{criterion}}}{0.9126\frac{G}{\rho}}\right\}. \tag{4.23}$$

It can immediately be seen in (4.23) that the static criterion is the critical one when the density of the wall ρ is low. The cross-over point occurs at $\rho = 0.4563$. For $\rho = 1$, the free-wave criterion is critical; but with high damping (so that $\nu > 0$), the

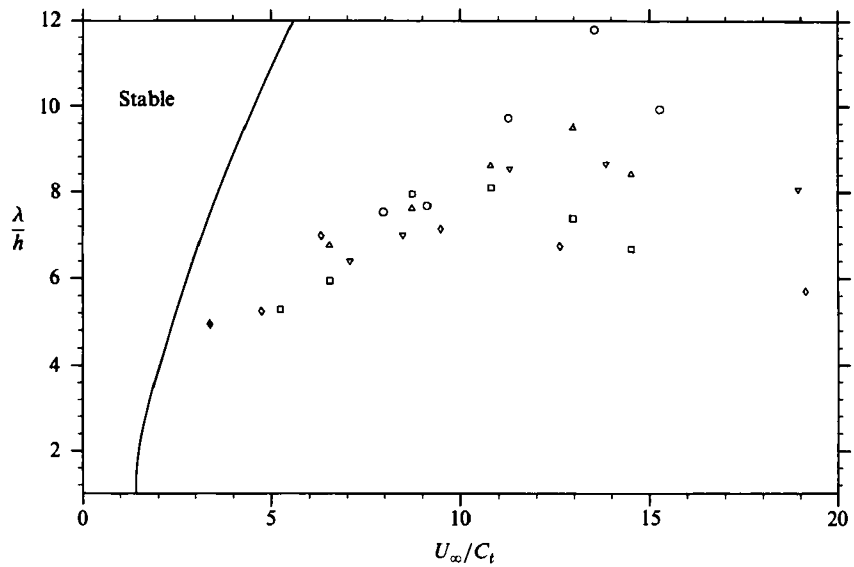

FIGURE 4. Comparison of the prediction of the static criterion and experimental static divergence results. ——, static criterion. Gad-el-Hak *et al.* (1984); ○, $h = 0.15$; △, $h = 0.24$; ▽, $h = 0.32$; □, $h = 0.40$; ◇, $h = 0.71$. Hansen *et al.* (1980): ◆, $h = 0.335$. h is in cm.

associated instability can be suppressed (see equation (4.11*b*)). For sufficiently damped viscoelastic walls, the important criterion tends to be the static criterion. However, this can also be resisted if $C_t > 0.7071\,U_\infty$. Thus, for $\rho = 1$, a sufficiently damped viscoelastic single-layer wall with $C_t > 0.7071\,U_\infty$ should not suffer from inviscid instability according to the stability criterion (4.23). A bound on U_∞ identical to our static criterion was given by Hansen & Hunston (1974). However, their analysis was more intuitive than rigorous. The importance of the free-wave bound $U_\infty < c_R$ for the suppression of compliance-related instability was in fact noted by Nonweiler (1962) in his study of the stability of laminar boundary-layer flow over a non-dissipative layer. Much more recently, the static and free-wave bounds on U_∞ given by (4.23) were also found by Fraser & Carpenter (1985) to give approximate critical velocities for inviscid flows over non-dissipative single-layer walls. As in §4.3 the imposition of zero-dissipation greatly restricts the possible values of c which Class B marginal states can have.

Finally we compare the predictions of the static and the free-wave criteria in the form of (4.17) (with α-dependence) with the experimentally observed occurrences of instabilities in turbulent boundary-layer flows over viscoelastic and nearly elastic single-layer compliant walls. The most comprehensive and well-documented sources of experimental results appear to be those of Gad-el-Hak *et al.* (1984) for static divergence instability (Class A) on viscoelastic layers and Gad-el-Hak (1986) for Class B instability on nearly elastic layers. Although the present criteria have been concocted in an essentially laminar-flow context, turbulent boundary-layer flows do in a fairly gross fashion share common features with thin laminar boundary-layer flow which is what they are in the mean.

Figure 4 shows the boundary for stability based on the static criterion $U_\infty^2 < E_\alpha^{(n)}/\alpha$ and the data points based on the experimentally observed occurrences of static divergence instability in Gad-el-Hak *et al.* (1984) for viscoelastic layers. One of the

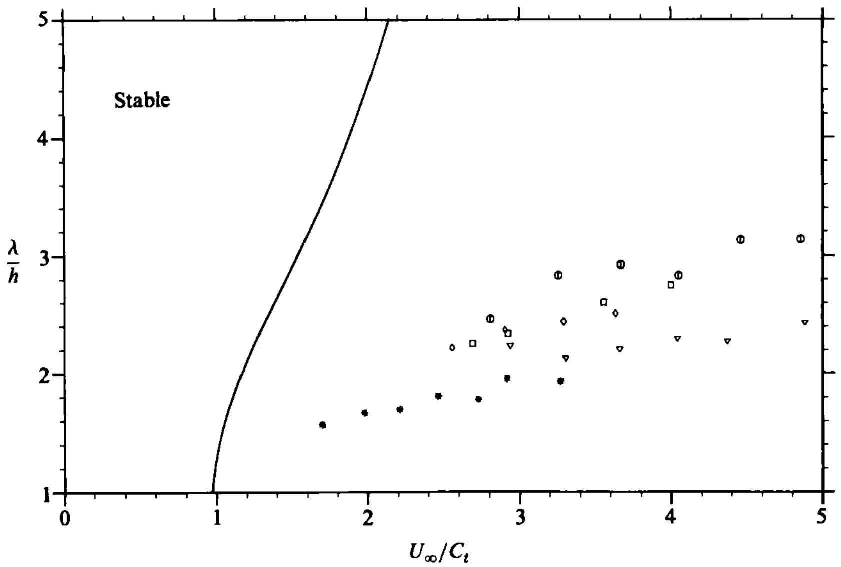

FIGURE 5. Comparison of the prediction of the free-wave criterion and the experimental Class B instability results. ——, free-wave criterion. Gad-el-Hak (1968): ∇, $h = 0.32$; $\square$, $h = 0.40$; $\oplus$, $h = 0.48$; $\diamond$, $h = 0.65$; $*$, $h = 1.05$. h is in cm.

data points was deduced from the reported results of Hansen *et al.* (1980). It can be seen that none of the instability data points falls in the region where the static criterion is satisfied. These unstable states were observed to have real phase velocities c_r typically less than $0.05U_\infty$. The smallness of the phase velocities is consistent with the 'static' nature of the present criterion. These experimentally observed instabilities are believed to be of the Class A type. Figure 5 shows the boundary for stability based on the free-wave criterion $U_\infty^2 < c_\alpha^2$ and the data corresponding to Class B instability states observed by Gad-el-Hak (1986) on nearly elastic walls. The real phase velocities in this case fall mainly in the range of $0.25U_\infty$ to $0.5U_\infty$. The low levels of material damping and higher real phase velocities suggest that the observed instabilities are predominantly of the Class B type. Again we note that there are no observed instability states that violate the free-wave criterion. The materials used in these experiments have densities $\rho \approx 1$. A comparison of the two criteria in figures 4 and 5 over the available ranges of λ/h shown reveals that the free-wave criterion is the more critical condition. This seems to agree with figure 3 of Gad-el-Hak (1986) which shows that the nearly elastic walls, which can support the Class B waves, have lower onset speeds for instability. The Class B instabilities, which normally have higher phase velocities, did not seem to occur on the viscoelastic walls of Gad-el-Hak *et al.* (1984). This is probably because of the fairly high levels of damping present in those viscoelastic walls. The bounds provided by $E_\alpha^{(n)}/\alpha$ and c_α^2 generally fall with increasing αh. Thus an increase in wall thickness h has the effect of tightening the bounds on U_∞ for the same range of α or λ. This is consistent with the observations of Gad-el-Hak *et al.* (1984), Gad-el-Hak (1986) that the onset U_∞ for both types of instability falls with increase in wall thickness.

The criterion for stability given by (4.17) also applies to more complex compliant walls such as those composed of a number of layers of different viscoelastic layers; but the analyses required to implement the criterion will be correspondingly more complicated.

5. Conclusions

It was shown that some of the most important theorems, concerning the temporal stability of inviscid parallel flows over rigid walls, could be extended in modified form to passive compliant walls. These results were stated as Propositions 1, 2 and 3 in §3. Proposition 1 shows that the real phase speed c_r of an unstable wave mode must satisfy $U_L < c_r < U_U$; where U_U is the larger of the maximum flow speed of the basic velocity profile and zero, and U_L is the smaller of the minimum flow speed and zero. This result represents an extension of a similar result of Rayleigh. Proposition 2 states that the phase velocity c for an unstable wave mode lies within the semicircle having a radius $\frac{1}{2}(U_U - U_L)$ centred at $\frac{1}{2}(U_U + U_L)$ on the real axis of the complex c-plane, and is a modified extension of the semicircle theorem of Howard (1961). For basic velocity distributions which vanish at the wall, Proposition 3 states that the bound on the temporal amplification rates of unstable modes of Høiland (1953) also applies to passive compliant walls generally.

The proofs of the proposition rest upon the realization that the coupling condition at the flow–wall interface is intimately related to the transfer of work from the flow to the wall. The class of passive compliant walls for which the propositions hold can broadly be identified as those which admit disturbance travelling-wave solutions similar to those of the inviscid flow regime. The generalization to a large class of passive compliant walls is made possible by adopting a variational-Lagrangian formulation of the essential dynamics of the walls which enables the work transferred from the flow to the wall to be represented in a very general manner as given in (2.8). In this representation, the characteristics of the wall are determined by three real-valued integrals I, E and D. These are respectively related to the Kinetic energy integral, the Stored energy integral and the Dissipation integral of the disturbance mode in the wall. The proofs of the Propositions rely only on the positive-definite nature of these integrals. The first proposition, however, does not require E to be positive.

An equation which governs the marginal stability of thin shear flows over general passive compliant walls was derived. From this equation, a set of criteria for the marginal instability of thin shear flows over compliant walls similar to those discussed by Benjamin (1963), in the context of tensioned membrane, was obtained.

Based on the bounds on the real phase speed c_r prescribed by Proposition 1, sufficient conditions for the marginal stability of thin shear flows over passive compliant walls were derived. The criteria for stability are of two types; a static criterion and a free-wave criterion. Bounds on the free-stream velocity of thin shear flows over bending plates on elastic foundation were obtained using the general criteria for marginal stability. These are identical to the critical velocities given in Carpenter & Garrad (1986); obtained under slightly more restrictive conditions. Similar bounds were also derived for single-layer viscoelastic walls. The prediction of stability given by the general criteria also shows good consistency with the experimental results of Gad-el-Hak *et al.* (1984), Gad-el-Hak (1986) and Hansen *et al.* (1980) for the observed occurrences of instabilities in turbulent flows over single-layer elastic and viscoelastic walls.

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